Calculating the Revenue-Maximizing Excise Tax

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ABSTRACT

This methodological note explains how to calculate the revenue-maximizing excise tax rate for goods such as gasoline, beer, or cigarettes. It considers the cases of linear, logarithmic, and Box-Cox demand curves for a single good, as well as the situation where two substitutes are taxed. The methodology is applied to the demand for gasoline and diesel fuel in Madagascar, where it is shown that the current (1996) excise tax rates are significantly below their revenue-maximizing levels.

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Author's note. Thanks are due to George Kuo, whose careful reading of the document and useful comments are much appreciated.

TABLE OF CONTENTS

1 Introduction

In some countries, the tax rate on alcoholic beverages, tobacco products, or motor fuels may be so high that it exceeds the revenue-maximizing tax rate. This note outlines a methodology for determining the revenue-maximizing tax rate for several useful cases. The next three sections set out the theory and section 5 provides an illustrative application.

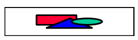
2 One good, infinitely elastic supply

The discussion begins with a single good and the assumption that the supply curve is horizontal, i.e., inifinitely elastic. This assumption is a reasonable approximation for most major excisable commodities.

2.1 Linear demand curve

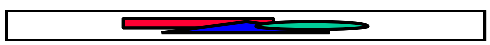
Figure 1 presents the case of a linear demand curve. The initial pretax price is P_0, at which price the quantity Q_0 is sold. Then a tax is imposed at rate t, the retail price rises to $P_0(1+t)$ and the quantity demanded falls to Q_1. The resulting tax revenue is denoted by the area *EFJG*. We need to calculate the tax rate t* that maximizes this area.

For a linear demand curve,

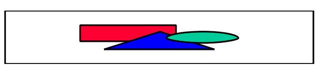


where $b < 0$ to give the characteristic downward slope of the demand curve. We also have

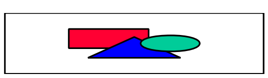
where $Q_1 = a + b(P_0(1+t))$. This implies that



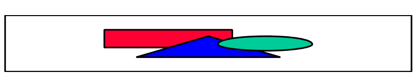
We then get the revenue-maximizing tax rate from



which yields

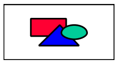
 (1)

An intuitive process leads to the same result (see Gamble 1989). The maximum revenue is given by the largest box, such as *BCHG*, which may be fitted into the triangle *AKG* (see Figure 1). The box is a square, with height $(A\text{-}G)/2$. From the equation of the demand curve, point A is given by $P = -a/b$ (b is a negative number). Therefore the revenue-maximizing tax *rate* is

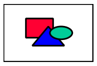


as in equation (1).

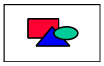
It is often helpful to express the equation in elasticity form, although the approximation is good only if η is relatively large (absolutely), i.e., if demand is relatively elastic (see Haughton, 1998, for further justification). Defining the own-price elasticity as



we have at point C (see Figure 1)



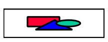
and therefore

 (2)

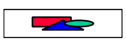
Strictly speaking, η should be measured at the point on the demand curve where there are no taxes (i.e., point C in Figure 1); in practice, it is typically measured at the observed tax rate (i.e., point F in Figure 1), which adds some further error to the approximation.

2.2 Constant elasticity demand curve

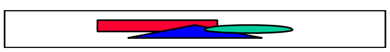
The second case focuses on a constant-elasticity demand curve, such as that shown in Figure 2. It may be written as



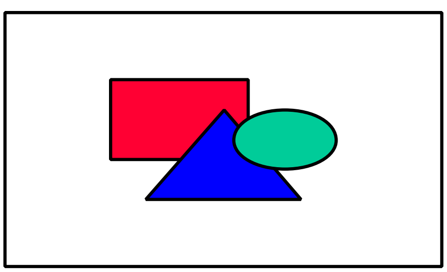
As before, we have



where $Q_1 = c[P_0(1+t)]^{\eta}$. This yields



To find the revenue-maximizing t^*, we have

 (3)

The result is elegant, but for reasons explained below, it is much less useful that either equations (1) or (2). Table 1 presents the revenue-maximizing tax rates that result from equations (2) and (3) for a selection of demand elasticities. The rates are much lower with the linear demand curve (equation (2)), indicating that the choice of the *form* of the demand curve is extremely important, particularly when discussing revenue-maximizing tax rates outside the known portion of the demand curve.

Table 1

Revenue-maximizing tax rates with different demand curves

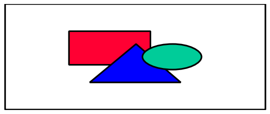
	Own-price elasticity of demand			
	-0.5	-1.0	-2.0	-5.0
Linear demand curve (equation (2))	100%	50%	25%	10%
Constant elasticity demand curve (equation (3))	NA	NA	100%	25%

When $\eta < -1$, which means elastic demand (e.g., $\eta = -2$), the results are sensible. But when $\eta = -1$, t^* is undefined, and when η -1, the revenue-maximizing tax rate is negative, which is nonsensical and in fact incorrect. When demand is sufficiently inelastic, the revenue-maximizing tax is in theory infinitely high; with a constant-elasticity demand curve a modicum of demand will exist even at extraordinarily high prices. Of course, such an outcome is not plausible; in other words, demand curves are not globally of constant elasticity with low (absolute) demand elasticities. Yet, at first sight, such curves look plausible, as shown by the two constant-elasticity demand curves graphed in Figure 3; the two curves have elasticities of -0.5 and -2.0, respectively, and are constructed so that when the price is 10, the quantity demanded is also 10. For a discussion of the estimation of demand curves, see Haughton (1998).

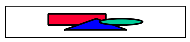
The practical problem is that the estimated values of the own-price elasticity of demand for the major excisable commodities are typically fairly small (absolutely). Indeed, one of the attractions of these goods as objects of taxation is that they typically face inelastic demand so that fairly high tax rates do not deter too many consumers.

2.3 The Box-Cox Transformation

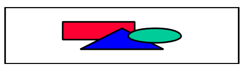
Demand curves are not necessarily either linear or constant-elasticity. One alternative sometimes used in the practical estimation of demand curves is the Box-Cox transformation. For variable *q*, this transformation is defined as



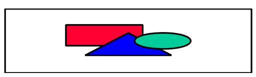
Thus, the demand curve could be written as

 (4)

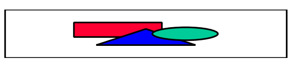
which reduces to the linear demand curve if $\gamma = \lambda = 1$ and to the constant-elasticity form if $a = \gamma = \lambda = 0$. Rewriting equation (4) in extensive form yields



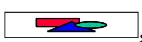
so



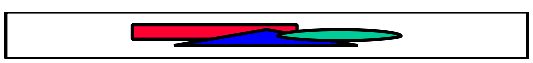
and hence



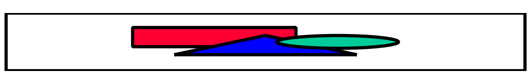
Given that revenue is expressed as

,

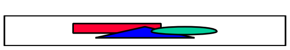
the result is



Maximizing gives



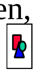
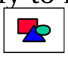
which simplifies to

 (5)

The only unknown is *t*, which may be solved by searching over a grid or by other techniques.

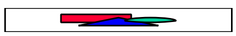
Substituting $\gamma = \lambda = 1$ in equation (5) yields the same expression as equation (2), which gives the revenue-maximizing tax rate for the linear case. And if $\gamma = \lambda = 0$, equation (5) reduces to equation (3), which is the appropriate formula for the case of a constant-elasticity demand curve.

3 Two Goods, Infinitely Elastic Supply

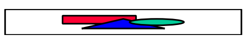
Often, we are interested in taxing two close substitutes, for instance, beer and stout (Guinness). The  that maximizes revenue (R_1) from beer alone (focusing on η_1, as in the previous section) is unlikely to maximize total revenue (R). In this case, the challenge is to determine the *pair* of tax rates  that maximize total revenue.

3.1 Linear Demand Curve

The case of two close substitutes is expressed as

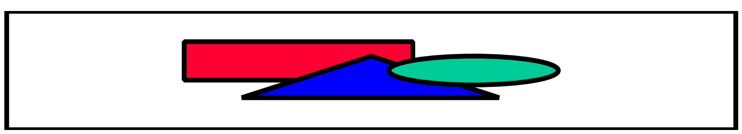


and

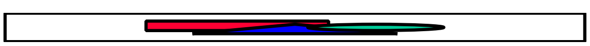


where Q_i is the quantity of good *i* demanded and P_i is the price of good $_i$. Total revenue is given by

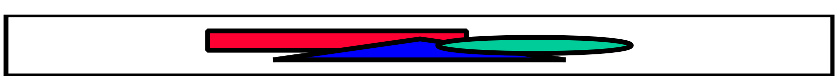
where P_{i} is the price of good $_i$ in the pretax situation. We then get



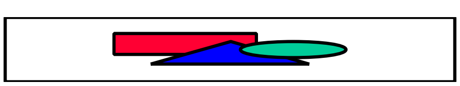
or, rearranging,



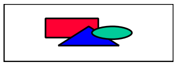
yielding

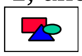
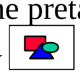
 (6)

where h_1 is the own-price elasticity of demand for good 1. Similarly, we have

 (7)

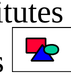
Substituting from equation (7) into (6) and rearranging, we get the solution

 (8)

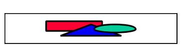
Example. Assume unit elastic demand for both goods, i.e., $h_1 = h_2 = -1$. Assume further that $b_1 = c_2 = -2$, that $b_2 = c_1 = 1$, and that the pretax prices are $P_1 = P_2 = \$10$. Substitution gives $a_1 = a_2 = B_1 = B_2 = 1/2$ and so  clearly  as well. In other words, the revenue-maximizing tax rates are 100% on each good. If, however, we had considered good 1 in isolation and applied equation (2), we would have concluded that the revenue-maximizing tax rate is just 50%. The key idea is that if only good *i* were taxed, and the only important substitute were another taxed good *j*, then the revenue-maximizing tax rate will be higher than if the only important substitute were an untaxed good. Stated another way, if good *j* is untaxed, a tax on good *i* will quickly push consumers to buy good *j*; this outcome is not as likely when good *j* is taxed as well.

For cigarettes and alcohol, the main alternatives to the taxed goods are typically untaxed goods that are found in the informal sector. Under these circumstances, the single-good case is appropriate (provided all taxable alcoholic beverages or all tobacco products are treated as a single unit). In the case of close taxed alternatives, however, as in the demand for gasoline, the two-good case is applicable. The situation can be extended to many substitutes, but at this point simple generalizations about revenue-maximizing tax rates are harder to make.

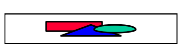
3.2 Constant-Elasticity Demand Curve

The case of two close substitutes does not lend itself to an analytic solution for the revenue-maximizing pair of tax rates , although some simplifications can lead to a single nonlinear equation that can be solved relatively easily. The algebraic details are given below for the truly interested, but as in the single-good case, this approach is less useful than the (locally more plausible) linear case.

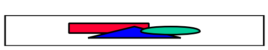
The demand curves are given by



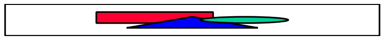
and



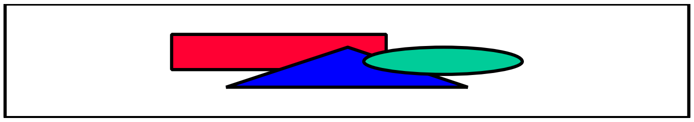
where the constant terms may hide other variables (e.g., income, a proxy for consumer tastes and preferences such as age, and so on). Revenue is given by



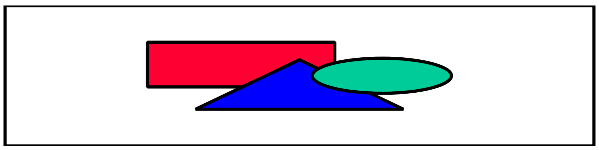
where



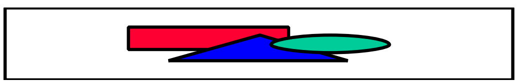
The first-order conditions for a maximum are given by

 (9)

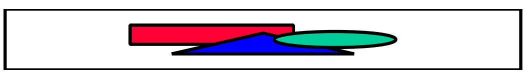
and similarly for $\delta R/\delta t_2$. By moving the second terms of these equations to the right-hand side and taking the ratio, we get



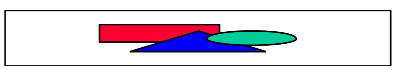
which with simplification yields



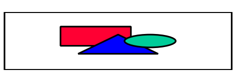
which reduces to



and yields



Further simplification gives

 (10)

Now t_1 from equation (10) can be substituted into equation (9). The only unknown in the new equation (9) is t_2, and its optimal value can be found by applying an optimization program (or by searching over a grid of values for t_2). Equation (10) is applicable only if $h_{ii} < -1$, which is rare; and even $h_{ii} < -1$ is only a necessary and not a sufficient condition for a solution.

4 One Good, Supply Not Infinitely Elastic

Figure 4 presents the case of one good with a supply that is not infinitely elastic (for linear demand and supply). It differs from Figure 1 in that the supply curve is upward sloping, which is the more conventional textbook case. In practice, however, the supply curve is usually assumed to be horizontal for the main excisable commodities.With the exception of agricultural commodities, demand curves have been estimated far more commonly than supply curves, with some of the results to be found in Glenday and Haughton (1992).

4.1 Linear Demand Curve

The more straightforward, and probably more plausible case in practice is the linear demand curve. Using the superscripts *d* for demand and *s* for supply, and ignoring other influences, we have

where $b < 0$ as usual, and

Revenue is given by

When there is no tax, $P^d = P^s$. With the tax wedge, however, $P^d = (1+t)P^s$. Given that the quantity demanded equals the quantity supplied in equilibrium, we introduce the tax and have

which, with rearrangement, yields

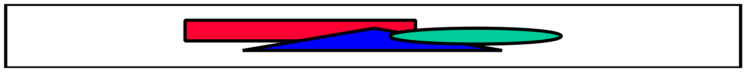
For any nontrivial situation, $c < a$. In other words, the supply curve starts below the demand curve such that an equilibrium exists for a positive value of output. Substituting into the supply curve gives

Revenue is now given by

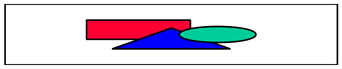
(11)

The first-order condition for a maximum is

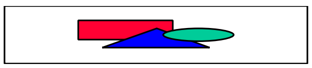
Simplification yields



and then

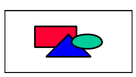


Further manipulation eventually yields

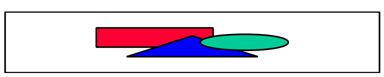
 (12)

4.2 Constant-Elasticity Curves

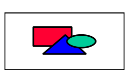
The case of constant-elasticity curves is only slightly less inelegant. Figure 5 presents the relevant diagram. We have



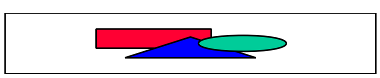
so that



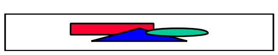
Similarly,



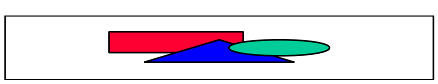
so that



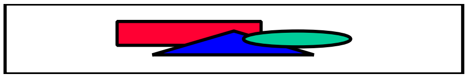
Using Figure 5, we see that



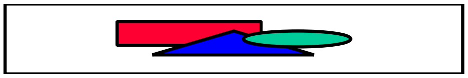
so that



and gives

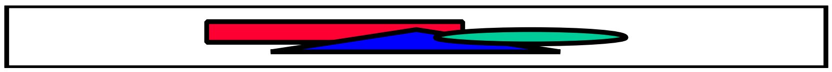


We therefore get

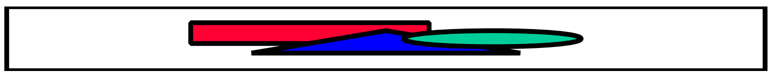


which will be needed below. The next step is to find the tax rate that maximizes revenue. We have

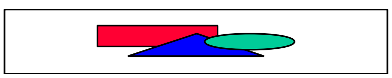
which, with substitution, gives



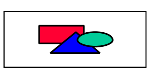
Maximizing gives



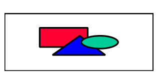
or, with simplification,

 (13)

For relatively low tax rates (below about 50%) and small demand elasticities (not below about -0.5), the third term in this equation is sufficiently small to ignore, in which case a little further manipulation yields the approximation

 (14)

More generally equation (13), which is nonlinear in *t*, can be solved for *t* when values are available for the elasticities. Using equation (13) one also gets



which is equation (2).

5 An Application

A companion methodological note (Haughton 1998) shows how demand curves may be estimated. The approach taken there was illustrated by applying it to the estimation of the demand curve for regular gasoline in Madagascar based on annual data for the period 1978--

1996. That note found that the most satisfactory estimates of demand elasticities came from applying a partial adjustment model, which yielded the following results:

Table 2

Elasticities of demand for regular gasoline in Madagascar

	Short-run	Long-run
Gasoline demand elasticity w.r.t. price of gasoline (h_{11})	-0.26	-0.93
Gasoline demand elasticity w.r.t. price of diesel fuel (h_{12})	0.08	0.28

Source: Methodological Note No. 2, Table 1.

For the calculations below, we also need the following information:

Table 3

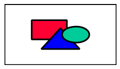
Prices and quantities for motor fuel in Madagascar

	post-tax	pre-tax
Index of real price of regular gasoline	7.2377	5.8891
Index of real price of diesel fuel	3.9178	3.4261
Index of quantity of regular gasoline consumed per capita	8,065.39	
Index of quantity of diesel fuel consumed per capita	17,667.71	

Source: Andrianomanana and Razafindravonona (1997).

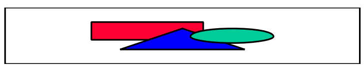
5.1 Applying the single-good equations

To recap, equation (1) gives

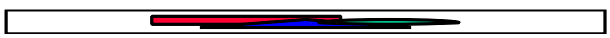


But, for the moment, we have an estimated own-price elasticity of demand of -0.93 for regular gasoline in Madagascar. The methodology, however, allows one to reconstitute a linear equation for application to the formula in equation (1).

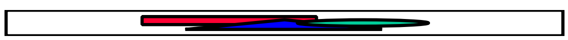
The demand curve is given by $Q = a + bP$ so that $b = dQ/dP$. We also have $\eta = (dQ/dP)(P/Q) = bP/Q$ so that $b = \eta Q/P$. By estimating demand at the mean observed price and quantity (i.e., P_1 and Q_1), we get



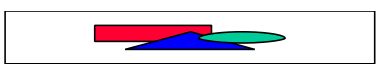
We also have



We know that the pretax price ($=P_0$) equals 5.8891. Hence,

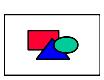


By applying equation (1), we thus get



In other words, the revenue-maximizing tax rate on regular gasoline in Madagascar is 78% of the pretax price, compared with the actual tax rate of 39% in 1996. This result assumes that the demand for regular gasoline has no important substitutes, which is not a reasonable assumption as demonstrated below.

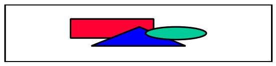
Equation (1) may be approximated by equation (2), which is given by



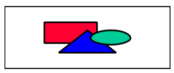
Substituting $\eta = -0.93$ gives $t^* = 0.54$, i.e., a tax rate of 54%. The approximation given in equation (3) is not applicable because the observed demand elasticity of -0.93 is not less than -1.

5.2 Applying the two-good equations

In practice, gasoline is a close substitute for diesel fuel in Madagascar, and it would be unwise to consider the revenue-maximizing tax on gasoline in isolation from the tax on diesel fuel. Using a similar approach to that taken for gasoline, we find that the long-run own-price elasticity of demand for diesel fuel is -1.06. The elasticity of gasoline demand with respect to the price of diesel fuel was found to be 0.28, and symmetry is applied to give the corresponding elasticity of diesel demand with respect to the price of gasoline. Using the terminology of section 3.1 we therefore have

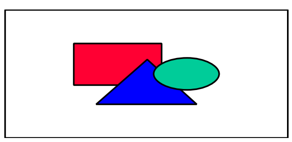


We wish to apply equations (8) and (7), which state



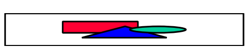
and

where

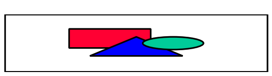


and α_2 and B_2 are defined in similar fashion. Here, $P_{1,0}$ is the pretax price of good 1; when the tax is imposed, the price rises to $P_{1,1}$. When estimating a demand curve, one has observations based on the price inclusive of tax, i.e., on $P_{1,1}$.

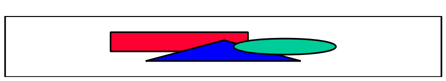
We now need to find the values of a_i, b_i, and c_i from the demand equations of the form



As in the previous section, we have

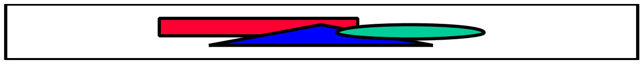


We also have

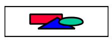


and

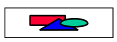
Thus,



Similar though tedious calculations give $\alpha 2 = 0.5830$, $B1 = 0.3475$ and $B2 = 0.2210$. By substituting into equation (8), we get

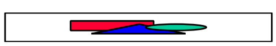


and, using equation (7), we have

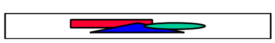


In other words, the revenue-maximizing tax rates appear to be about 104% for regular gasoline and 81% for diesel fuel. If we had focused only on the gasoline market and used equation (1) we would have found a revenue-maximizing tax rate of 78% for regular gasoline. We would also have implicitly ignored the revenue effects of buying a substitute such as diesel fuel. In reality, if both gasoline and diesel fuel were taxed, the revenue-maximizing rate on both would be higher than if only one commodity were taxed. To see why, suppose that the tax rate on gasoline were raised; an increase would have a direct effect on the tax collected from gasoline, but it would also push some consumers to switch to diesel fuel. But if diesel fuel were taxed too, the shift away from gasoline would not harm total government revenue as much as if diesel fuel had been untaxed.

The above illustration looks at the long-run elasticities of demand. In the short-run, demand for gasoline is more inelastic. Accordingly (from Table 1 in Methodological Note 2) we arrive at



for gasoline, and



for diesel fuel. With these parameters, the revenue-maximizing tax rate on gasoline in the single-good case is 246%. In the two-good case, the revenue-maximizing tax rates are 319% on gasoline and 227% on diesel fuel. In the short-run, which is about a year in this case, very high tax rates would yield substantial revenue; but, over time, people shift away from consuming fuels, such that the use of the long-run elasticities (with their associated lower maximum tax rates) becomes appropriate. Furthermore, with high rates, evasion and smuggling are likely to occur, making the even the above calculations here somewhat unrealistic.

As of 1996, the tax rate on regular gasoline in Madagascar was 39% and the rate on diesel fuel 24%. These are well below the revenue-maximizing rates, but are not necessarily too low. It does mean, however, that higher tax rates on these fuel types would yield more revenue than is currently the case.[1]

[1] In passing one might note that Merriman (1994) found that taxes on cigarettes in the United States are also well below the revenue-maximizing level.

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Appendix

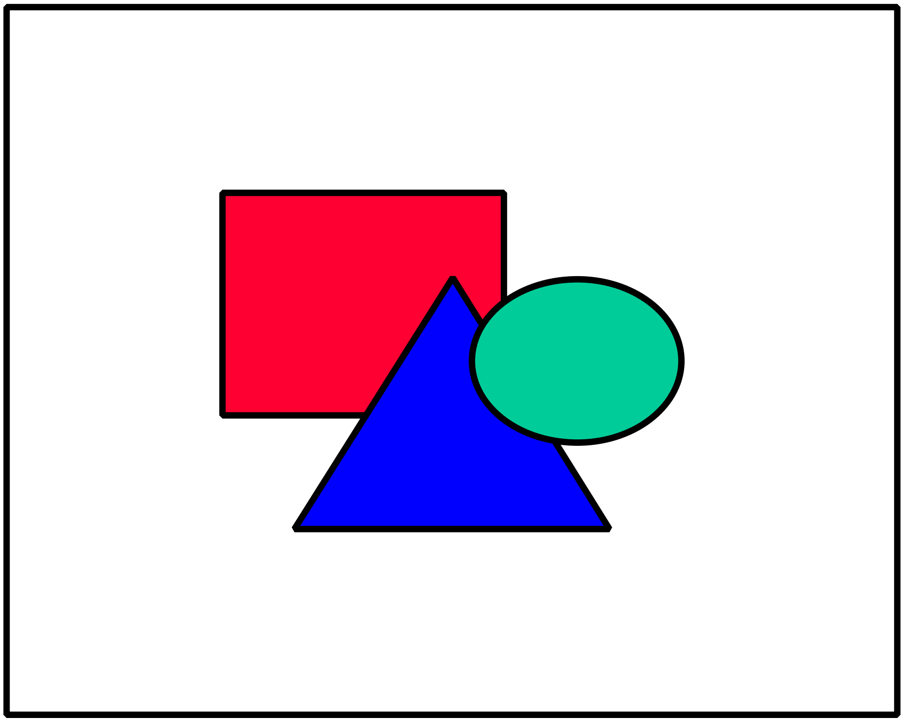

Figure 1

Linear Demand Curve with Infinitely Elastic Supply

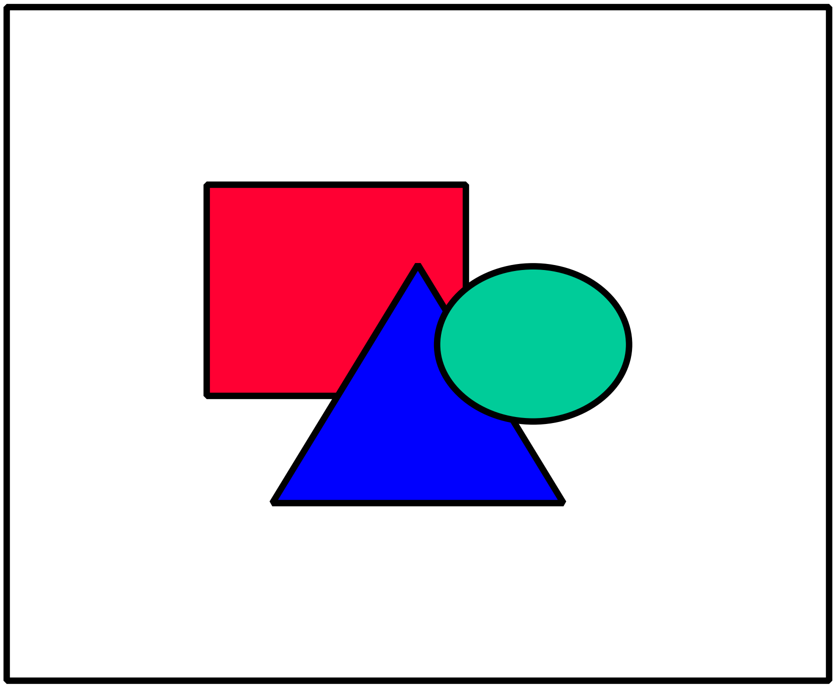

Figure 2

Constant-Elasticity Demand Curve with Infinitely Elastic Supply

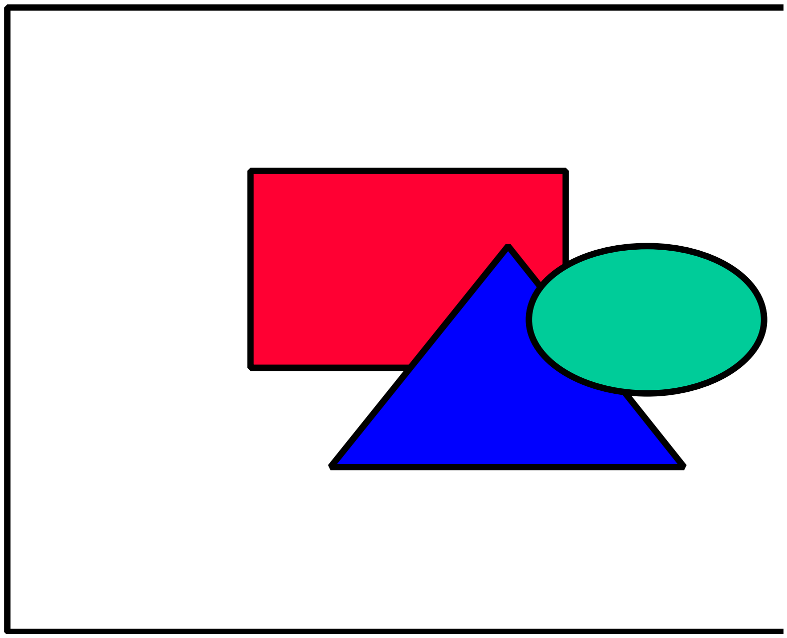

Figure 3

Constant-Elasticity Demand Curves

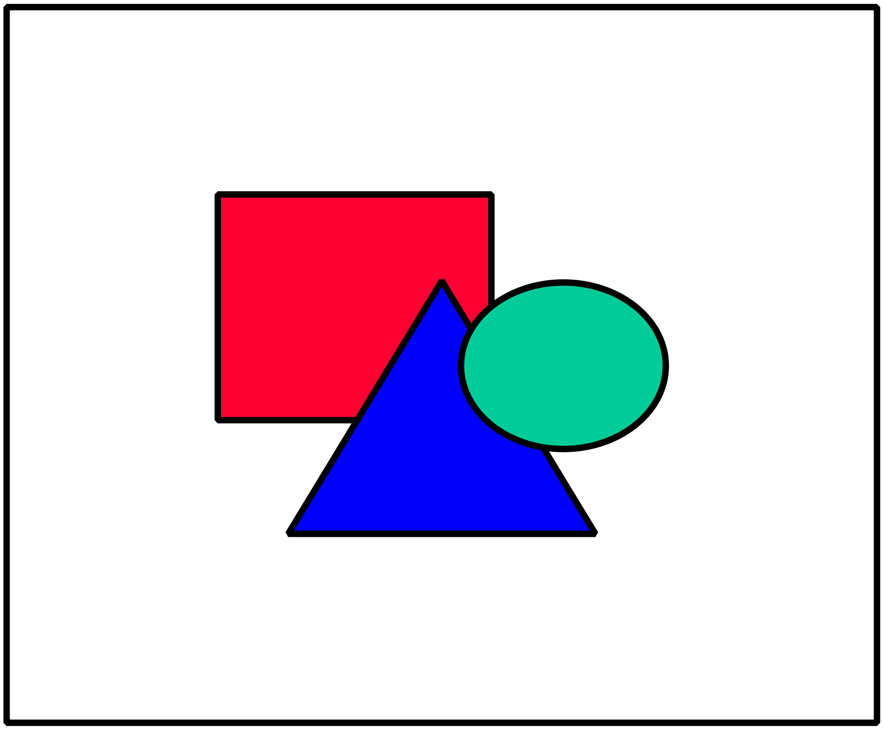

Figure 4

Linear Demand and Supply Curves: The General One-Good

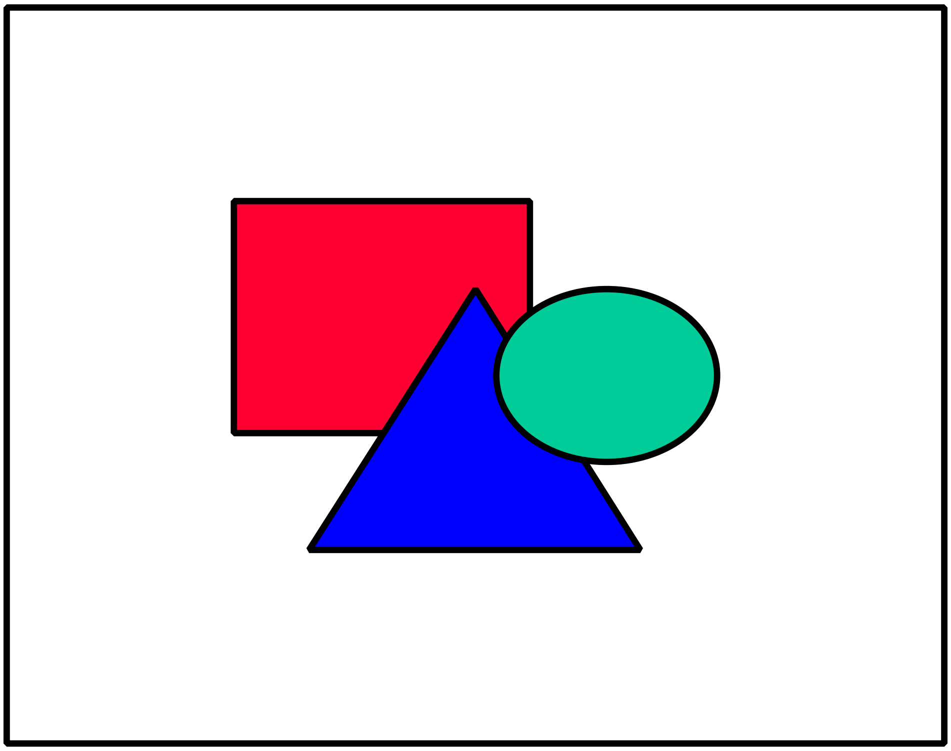

Figure 5

Constant-Elasticity Demand and Linear Supply Curves: The General One-Good Case